

# THE LIMIT OF A FUNCTION

Let us investigate the behavior of the function  $f$  defined by  $f(x) = x^2 - x + 2$  for values near 2. The following table gives values of  $f(x)$  for values of  $x$  close to 2 but not equal to 2.

$x$	$f(x)$	$x$	$f(x)$
1, 0	2, 000000	3, 0	8, 000000
1, 5	2, 750000	2, 5	5, 750000
1, 8	3, 440000	2, 2	4, 640000
1, 9	3, 710000	2, 1	4, 310000
1, 95	3, 852500	2, 05	4, 152500
1, 99	3, 970100	2, 01	4, 030100
1, 995	3, 985025	2, 005	4, 015025
1, 999	3, 997001	2, 001	4, 003001

From the table and the graph of  $f$  (a parabola) shown in Figure 1 we see that when  $x$  is close to 2 (on either side of 2)  $f(x)$  is close to 4. In fact, it appears that we can make the values of  $f(x)$  as close as we like to 4 by taking  $x$  sufficiently close to 2. We express this by saying "the limit of the function  $f(x) = x^2 - x + 2$  as  $x$  approaches 2 is equal to 4." The notation for this is

$$\lim_{x \rightarrow 2} (x^2 - x + 2) = 4$$

In general, we use the following notation.

DEFINITION 1 We write

$$\lim_{x \rightarrow a} f(x) = L$$

and say "the limit of  $f(x)$ , as  $x$  approaches  $a$ , equals  $L$ " if we can make the values of  $f(x)$  arbitrarily close to  $L$  (as close to  $L$  as we like) by taking  $x$  to be sufficiently close to  $a$  but not equal to  $a$ .

Roughly speaking, this says that the values of  $f(x)$  get closer and closer to the number  $L$  as  $x$  get closer and closer to the number  $a$  (from either side of  $a$ ) but  $x \neq a$ . An alternative notation for

$$\lim_{x \rightarrow a} f(x) = L \text{ is } f(x) \rightarrow L \text{ as } x \rightarrow a$$

which is usually read "  $f(x)$  approaches  $L$  as  $x$  approaches  $a$ ."

Notice the phrase "but  $x \neq a$ " in the definition of limit. This means that in finding the limit of  $f(x)$  as  $x$  approaches  $a$ , we never consider  $x = a$ . In fact  $f(x)$  need not even be defined when  $x = a$ . The only thing that matters is how  $f$  is defined near  $a$ . Figure 2 shows the graphs of three functions. Note that

in part (c)  $f(a)$  is not defined and in part (b),  $f(a) \neq L$ . But in each case, regardless of what happens at  $a$ ,  $\lim_{x \rightarrow a} f(x) = L$ .

EXAMPLE Guess the value of  $\lim_{x \rightarrow 1} \frac{x-1}{x^2-1}$ .

SOLUTION Notice that the function  $f(x) = \frac{x-1}{x^2-1}$  is not defined when  $x = 1$ , but that doesn't matter because the definition of  $\lim_{x \rightarrow a} f(x)$  says that we consider values of  $x$  that are close to  $a$  but not equal to  $a$ . The following table gives values of  $f(x)$  (correct to six decimal places) for values of  $x$  that approach 1 (but are not equal to 1).

$x$	$f(x)$	$x$	$f(x)$
0,5	0,666667	1,5	0,400000
0,9	0,526316	1,1	0,476190
0,99	0.502513	1,01	0.497512
0,999	0,500250	1,001	0,499750
0,9999	0,500025	1,0001	0,499975

On the basis of the values in the table, we make the guess that  $\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = \frac{1}{2}$ . This example is illustrated by the graph of  $f$  in Figure 3. Now let us change  $f$  slightly by giving it the value 2 when  $x = 1$  and calling the resulting function  $g$ :

$$g(x) = \begin{cases} \frac{x-1}{x^2-1}, & \text{if } x \neq 1 \\ 2, & \text{if } x = 1 \end{cases}$$

This new function  $g$  still has the same limit as  $x$  approaches 1 (see Figure 4).

EXAMPLE Find  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ .

SOLUTION Again the function  $f(x) = \frac{\sin x}{x}$  is not defined when  $x = 0$ . Using a calculator (and remembering that, if  $x \in \mathcal{R}$ ,  $\sin x$  means the sine of the angle whose *radian* measure is  $x$ ), we construct the accompanying table of values correct to eight decimal places. From the table and Figure 5 (drawn with the aid of the table) we guess that

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$x$	$\frac{\sin x}{x}$
$\pm 1.0$	0.84147098
$\pm 0.5$	0.95885108
$\pm 0.4$	0.97354586
$\pm 0.3$	0.98506736
$\pm 0.2$	0.99334665
$\pm 0.1$	0.99833417
$\pm 0.05$	0.99958339
$\pm 0.01$	0.99998333
$\pm 0.005$	0.99999583
$\pm 0.001$	0,99999983

This guess is in fact correct, as will be proved in another Chapter using a geometric argument.

EXAMPLE Find  $\lim_{x \rightarrow 0} \sin \frac{\pi}{x}$ .

SOLUTION Once again the function  $f(x) = \sin(\pi/x)$  is undefined at 0. Evaluating the function for some small values of  $x$ , we get

$$\begin{aligned} f(1) &= \sin \pi = 0 & f\left(\frac{1}{2}\right) &= \sin 2\pi = 0 \\ f\left(\frac{1}{3}\right) &= \sin 3\pi = 0 & f\left(\frac{1}{4}\right) &= \sin 4\pi = 0 \\ f(0.1) &= \sin 10\pi = 0 & f(0.01) &= \sin 100\pi = 0 \end{aligned}$$

Similarly,  $f(0.001) = f(0.0001) = 0$ . On the basis of this information we might be tempted to guess that

$$\lim_{x \rightarrow 0} \sin \frac{\pi}{x} = 0$$

but this time our guess is wrong. Note that although  $f\left(\frac{1}{n}\right) = \sin n\pi = 0$  for any integer  $n$ , it is also true that  $f(x) = 1$  for infinitely many values of  $x$  that approach 0. In fact,  $\sin\left(\frac{\pi}{x}\right) = 1$  when  $\frac{\pi}{x} = \frac{\pi}{2} + 2n\pi$  and solving for  $x$ , we get  $x = \frac{2}{4n+1}$ . The graph of  $f$  is given in Figure 6. The broken lines indicate that the values of  $\sin\left(\frac{\pi}{x}\right)$  oscillate between 1 and -1 infinitely often as  $x$  approaches 0. Since the values of  $f(x)$  do not approach a fixed number as  $x$  approaches 0,

$$\lim_{x \rightarrow 0} \sin\left(\frac{\pi}{x}\right) \text{ does not exist.}$$

EXAMPLE The Heaviside function  $H$  is defined by  $H(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0 \end{cases}$

This function is named after the electrical engineer Oliver Heaviside (1850-1925) and can be used to describe an electric current that is switched on at time  $t = 0$ . Its graph is shown in Figure 7.

As  $t$  approaches 0 from the left,  $H(t)$  approaches 0. As  $t$  approaches 0 from the right,  $H(t)$  approaches 1. There is no single number that  $H(t)$  approaches as  $t$  approaches 0. Therefore  $\lim_{t \rightarrow 0} H(t)$  does not exist.

### ONE- SIDED LIMITS

In the last example we noticed that  $H(t)$  approaches 0 as  $t$  approaches 0 from the left and  $H(t)$  approaches 1 as  $t$  approaches 0 from the right. We indicate this situation symbolically by writing

$$\lim_{t \rightarrow 0^-} H(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow 0^+} H(t) = 1$$

The symbol " $t \rightarrow 0^-$ " indicates that we consider only values of  $t$  that are less than 0. Likewise " $t \rightarrow 0^+$ " indicates that we consider only values of  $t$  that are greater than 0.

DEFINITION 2 We write  $\lim_{x \rightarrow a^-} f(x) = L$  and say the left-hand limit of  $f(x)$  as  $x$  approaches a  $a$  [ or the limit of  $f(x)$  as  $x$  approaches a from the left] is equal to  $L$  if we can make the values of  $f(x)$  arbitrarily close to  $L$  by taking  $x$  to be sufficiently close to  $a$  and  $x$  less than  $a$ .

Notice that Definition 2 differs from Definition 1 only in that we require  $x$  to be less than  $a$ . Similarly, if we require that  $x$  be greater than  $a$ , we get "the right-hand limit of  $f(x)$  as  $x$  approaches is equal to  $L$  and we write

$$\lim_{x \rightarrow a^+} f(x) = L$$

Thus the symbol " $x \rightarrow a^+$ " means that we consider only  $x > a$ .

By comparing Definition 1 with the definition of one-sided limits, we see that the following is true.

DEF. 3  $\lim_{x \rightarrow a} f(x) = L$  if and only if  $\lim_{x \rightarrow a^-} f(x) = L$  and  $\lim_{x \rightarrow a^+} f(x) = L$

EXAMPLE The graph of a function  $g$  is shown in Figure 8. Use it to state the values (if they exist) of the following:

- (a)  $\lim_{x \rightarrow 2^-} g(x)$  (b)  $\lim_{x \rightarrow 2^+} g(x)$  (c)  $\lim_{x \rightarrow 2} g(x)$  (d)  $g(2)$   
 (e)  $\lim_{x \rightarrow 5^-} g(x)$  (f)  $\lim_{x \rightarrow 5^+} g(x)$  (g)  $\lim_{x \rightarrow 5} g(x)$  (h)  $g(5)$

SOLUTION

From the graph we see that (a)  $\lim_{x \rightarrow 2^-} g(x) = 3$  and (b)  $\lim_{x \rightarrow 2^+} g(x) = 1$

(c) Since the left and right limits are different, we conclude that  $\lim_{x \rightarrow 2} g(x)$  does not exist. (d)  $g(2)$  does not exist.

The graph also shows that (e)  $\lim_{x \rightarrow 5^-} g(x) = 2$  and (f)  $\lim_{x \rightarrow 5^+} g(x) = 2$ .

(g)  $\lim_{x \rightarrow 5} g(x) = 2$ . (h) Despite this fact, notice  $g(5) \neq 2$  and  $g(5) = 1$ .

INFINITE LIMITS

EXAMPLE Find  $\lim_{x \rightarrow 0} \frac{1}{x^2}$  if it exists.

SOLUTION As  $x$  becomes close to 0,  $x^2$  also becomes close to 0, and  $\frac{1}{x^2}$  becomes very large. (See the table.)

$x$	$\frac{1}{x^2}$
$\pm 1$	1
$\pm 0.5$	4
$\pm 0.2$	25
$\pm 0.1$	100
$\pm 0.05$	400
$\pm 0.01$	10000
$\pm 0.001$	1000000

In fact, it appears from the graph of the function  $f(x) = \frac{1}{x^2}$  shown in F.9 that the values of  $f(x)$  can be made arbitrarily large by taking  $x$  close enough to 0. Thus the values of  $f(x)$  do not approach a number, so  $\lim_{x \rightarrow 0} (\frac{1}{x^2})$  does not exist. To indicate the kind of behavior exhibited in EXAMPLE we use the notation

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$

This does not mean that we are regarding  $\infty$  as a number. Nor does it mean that the limit exists. It simply expresses the particular way in which the limit does not exist:  $\frac{1}{x^2}$  can be made as large as we like by taking  $x$  close enough to 0. In general, we write symbolically

$$\lim_{x \rightarrow a} f(x) = \infty$$

to indicate that the values of  $f(x)$  become larger and larger (or "increase without bound") as  $x$  becomes closer and closer to  $a$ .

**DEFINITION 4** Let  $f$  be a function defined on both sides of  $a$ , except possibly at  $a$  itself. Then

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that the values of  $f(x)$  can be made arbitrarily large ( as large as we please) by taking  $x$  sufficiently close to  $a$  ( but not equal to  $a$ ).

Another notation for  $\lim_{x \rightarrow a} f(x) = \infty$  is  $f(x) \rightarrow \infty$  as  $x \rightarrow a$

Again, the symbol  $\infty$  is not number , but the expression  $\lim_{x \rightarrow a} f(x) = \infty$  is often read as " the limit of  $f(x)$ , as  $x$  approaches  $a$ , is infinity"

or "  $f(x)$  becomes infinite as  $x$  approaches  $a$ "

or "  $f(x)$  increases without bound as  $x$  approaches  $a$ ".(See Figure 10.)

A similar sort of limit, for function that become large negative as  $x$  gets close to  $a$ , is defined in Definition 5.(See Figure 11.)

**DEFINITION 5** Let  $f$  be a function defined on both sides of  $a$ , except possibly at  $a$  itself. Then

$$\lim_{x \rightarrow a} f(x) = -\infty$$

means that the values of  $f(x)$  can be made arbitrarily large negative( as large as we please) by taking  $x$  sufficiently close to  $a$  ( but not equal to  $a$ ).

The symbol  $\lim_{x \rightarrow a} f(x) = -\infty$  can be read as " the limit of  $f(x)$ , as  $x$  approaches  $a$ , is negative infinity" or "  $f(x)$  decreases without bound as  $x$  approaches  $a$ ."

As an example we have  $\lim_{x \rightarrow 0} \left(-\frac{1}{x^2}\right) = -\infty$ .

Similar definitions can be given for the one-sided infinite limits

$$\begin{array}{ll} \lim_{x \rightarrow a^-} f(x) = \infty & \lim_{x \rightarrow a^+} f(x) = \infty \\ \lim_{x \rightarrow a^-} f(x) = -\infty & \lim_{x \rightarrow a^+} f(x) = -\infty \end{array}$$

remembering that " $x \rightarrow a^-$ " means that we consider only values of  $x$  that are less than  $a$ , and similarly " $x \rightarrow a^+$ " means that we consider only values of  $x$  that are larger than  $a$ .

Illustrations of these four cases are given in Figure 12.

### EXERCISES

- For the function  $f$  whose graph is given, state the value of the given quantity, if it exists.(Graph is given in Figure 13.)

$$\begin{array}{lll} (a) \lim_{x \rightarrow 1} f(x) & (b) \lim_{x \rightarrow 3^-} f(x) & (c) \lim_{x \rightarrow 3^+} f(x) \\ (d) \lim_{x \rightarrow 3} f(x) & (e) f(3) & (f) \lim_{x \rightarrow -2^-} f(x) \\ (g) \lim_{x \rightarrow -2^+} f(x) & (h) \lim_{x \rightarrow -2} f(x) & (i) f(-2) \end{array}$$

- For the function  $g$  whose graph is given, state the value of the given quantity, if it exists.(Graph is given in Figure 14.)

$$\begin{array}{llll} (a) \lim_{x \rightarrow -2^-} g(x) & (b) \lim_{x \rightarrow -2^+} g(x) & (c) \lim_{x \rightarrow -2} g(x) & (d) g(-2) \\ (e) \lim_{x \rightarrow 2^-} g(x) & (f) \lim_{x \rightarrow 2^+} g(x) & (g) \lim_{x \rightarrow 2} g(x) & (h) g(2) \\ (i) \lim_{x \rightarrow 4^+} g(x) & (j) \lim_{x \rightarrow 4^-} g(x) & (k) g(0) & (l) \lim_{x \rightarrow 0} g(x) \end{array}$$

3. State the value of the limit, if it exists, from the given graph. (Graph is given in Figure 15.)

$$\begin{array}{lll} (a) \lim_{x \rightarrow 3} f(x) & (b) \lim_{x \rightarrow 1} f(x) & (c) \lim_{x \rightarrow -3} f(x) \\ (d) \lim_{x \rightarrow 2^-} f(x) & (e) \lim_{x \rightarrow 2^+} f(x) & (f) \lim_{x \rightarrow 2} f(x) \end{array}$$

4. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount  $f(t)$  of the drug in the bloodstream after  $t$  hours. (Later we will be able to compute dosages and time intervals to ensure that the concentration of the drug does not reach harmful levels.)

Find  $\lim_{t \rightarrow 12^-} f(t)$  and  $\lim_{t \rightarrow 12^+} f(t)$  and explain the significance of these one-sided limits. (The graph is given in Figure 16.)

5. (a) Sketch the graph of the function

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(b) Use the graph from part (a) to state the value of each of the following limits, if it exists.

$$(i) \lim_{x \rightarrow 0^-} f(x) \quad (ii) \lim_{x \rightarrow 0^+} f(x) \quad (iii) \lim_{x \rightarrow 0} f(x)$$

6. (a) Sketch the graph of the function

$$f(x) = \begin{cases} 2 - x & \text{if } x < -1 \\ x & \text{if } -1 \leq x < 1 \\ 4 & \text{if } x = 1 \\ 4 - x & \text{if } x > 1 \end{cases}$$

(b) Use the graph from part (a) to state the value of each of the following limits, if it exists.

$$\begin{array}{lll} (i) \lim_{x \rightarrow -1^-} g(x) & (ii) \lim_{x \rightarrow -1^+} g(x) & (iii) \lim_{x \rightarrow -1} g(x) \\ (iv) \lim_{x \rightarrow 1^-} g(x) & (v) \lim_{x \rightarrow 1^+} g(x) & (vi) \lim_{x \rightarrow 1} g(x) \end{array}$$

7. Determine the infinite limit:

$$\begin{array}{lll} (a) \lim_{x \rightarrow 5^+} \frac{6}{x-5} & (b) \lim_{x \rightarrow 5^-} \frac{6}{x-5} & (c) \lim_{x \rightarrow 3} \frac{1}{(x-3)^8} \\ (d) \lim_{x \rightarrow 0} \frac{x-1}{x^2(x+2)} & (e) \lim_{x \rightarrow -2^+} \frac{x-1}{x^2(x+2)} & (f) \lim_{x \rightarrow \pi^-} \frac{1}{\sin x} \end{array}$$

THE LIMIT OF FUNCTION



