Combinations of functions

Two functions f and g can be combined to form new functions $f+g$, $f-g$, $f\cdot g$, $\frac{f}{g}$ in a manner similar to the way we add, subtract, multiply, and divide real numbers. If we define the sum $f+g$ by the equation

$$(f+g)(x)=f(x)+g(x)$$

then the right side of equation makes sense if both $f(x)$ and $g\,(x)$ are defined, that is, if x belongs to the domain of f and also to the of g. If the domain of f is A and the domain of g is B, then the domain of $f+g$ is the intersection of these domains, that is, $A\cap B$.
Notice that the + sign on the left side of equation stands for the operation of addition of *functions*, but the + sign on the right side of the equation stands for addition of the *numbers* $f(x)$ and $g(x)$.
Similarly, we can define the difference $f-g$ and the product fg, and their domains are also $A\cap B$. But in defining the quotient $\frac{f}{g}$ we must remember not to divide by 0.
EXAMPLE If $f(x)=\sqrt{x}$ and $g\,(x)=\sqrt{4-x^2}$, find the functions $f+g$, $f-g$, $f\cdot g$ and $\frac{f}{g}$.
SOLUTION The domain of $f(x)=\sqrt{x}$ is $[\,0,\infty)$. The domain of $g\,(x)=\sqrt{4-x^2}$ consist of all numbers x such that $4-x^2\geq 0$, that is, $x^2\leq 4$. Taking square roots of both sides, we get $\mid x\mid\leq 2$, or $-2\leq x\leq 2$, so the domain of g is the interval $[\,-2,2\,]$. The intersection of the domains of f and g is $[\,0,\infty)\cap[\,-2,2\,]=[\,0,2\,]$.
Thus, according to the definitions, we have
$(f+g)(x)=\sqrt{x}+\sqrt{4-x^2}$ $\quad 0\leq x\leq 2$
$(f-g)(x)=\sqrt{x}-\sqrt{4-x^2}$ $\quad 0\leq x\leq 2$
$(fg)(x)\quad=\sqrt{x}\sqrt{4-x^2}=\sqrt{4x-x^3}$ $\quad 0\leq x\leq 2$
$\left(\frac{f}{g}\right)(x)\quad=\quad\frac{\sqrt{x}}{\sqrt{4-x^2}}=\sqrt{\frac{x}{4-x^2}}$ $\quad 0\leq x<2.$
Notice that the domain of f/g is the interval $[\,0,2)$ because we must exclude the points where $g(x)=0$, that is , $x=\pm 2$.

COMPOSITION OF FUNCTION

There is another way of combining two function to get a new function. For example, suppose that $y=f(u)=\sqrt{u}$ and $u=g\,(x)=x^2+1$. Since y is a function of u and u is, in turn, a function of x, it follows that y is ultimately a function of x. We compute this by substitution :

$$y=f(u)=f(g(x))=f(x^2+1)=\sqrt{x^2+1}$$

The procedure is called *composition* because the new function is *composed* of the two given function f and g.
In general, given any two function f and g , we start with a number x in the domain of g and find its image $g\,(x)$. If this number $g\,(x)$ is in the domain of f, then we can calculate the value of $f(g\,(x))$. The result is a new function $h(x) = f(g\,(x))$ obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ (" f circle g ").
The domain of $f \circ g$ is the set of all x in the domain of g such that $g\,(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g\,(x)$ and $f(g\,(x))$ are defined.

EXAMPLE If $f(x) = x^2$ and $g\,(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$ and state their domains.

SOLUTION We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$$

The domains of both $f \circ g$ and $g \circ f$ are $\mathcal{R}$(the set of all real numbers).

NOTE : You can see from example that, in general, $f \circ g \neq g \circ f$. Remember, the notation $f \circ g$ means that the function g is applied first and then f is applied second. In example, $f \circ g$ is the function that first subtracts 3 and then squares; $g \circ f$ is the function that first sqaures and then subtracts 3.

EXAMPLE If $f(x) = \sqrt{x}$ and $g\,(x) = \sqrt{2 - x}$, find each and its domain :
(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$.

SOLUTION (a) $(f \circ g)(x) = f(g\,(x)) = f(\sqrt{2 - x}) = \sqrt{\sqrt{2 - x}} = \sqrt[4]{2 - x}$.
The domain of $f \circ g$ is $\{x \mid 2 - x \geq 0\} = \{x \mid x \leq 2\} = (-\infty, 2\,]$.
(b) $(g \circ f)(x) = g\,(f(x)) = g\,(\sqrt{x}) = \sqrt{2 - \sqrt{x}}$.
For $\sqrt{x}$ to be defined we must have $x \geq 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined we must have $2 - \sqrt{x} \geq 0$, that is , $\sqrt{x} \leq 2$, or $x \leq 4$. thus we have $0 \leq x \leq 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.
(c) $(f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$.
The domain of $f \circ f$ is $[\,0, \infty]$.
(d) $(g \circ g)(x) = g(g(x)) = g(\sqrt{2 - x}) = \sqrt{2 - \sqrt{2 - x}}$.
This expression is defined when $2 - x \geq 0$, that is, $x \leq 2$, and $2 - \sqrt{2 - x} \geq 0$. This latter inequality is equivalent to $\sqrt{2 - x} \leq 2$, or , $2 - x \leq 4$, that is $x \geq -2$. Thus $-2 \leq x \leq 2$, so the domain of $g \circ g$ is the closed interval $[\,-2, 2\,]$.

EXERCISES

1. Find the domain and range of the function :
   (a) $f(x) = 2x + 7,\ -1 \leq x \leq 6$; (b) $f(x) = 6 - x,\ -2 \leq x \leq 3$.
   (c) $g(x) = \frac{2}{3x - 5}$; (d) $h(x) = \sqrt{1 - x^2}$.

2. Find the domain of the function :
   (a) $f(x) = \frac{x^4}{x^2 + x - 6}$; (b) $g(x) = \sqrt{x^2 - 2x - 8}$;
   (c) $\phi(x) = \sqrt{\frac{x^2 - 2x}{x - 1}}$; (d) $\theta(x) = \sqrt[3]{x - 1}$.

3. Find the domain and sketch the graph of the function :
(a) $f(x) = x^2 + 2x - 1,\ \ g(x) = -x^2 + 6x - 7$;
(b) $f(x) = \sqrt{-x}$, $\qquad g(x) = \sqrt{6 - 2x}$;
(c) $h(x) = \sqrt{4 - x^2}$, $\qquad \phi(x) = \sqrt{x^2 - 4}$;
(d) $F(x) = \begin{cases} 2x + 3, & x < -1 \\ 3 - x, & x \geq -1 \end{cases}$; (e) $\zeta(x) = \begin{cases} x + 2, & x \leq -1 \\ x^2, & x > -1 \end{cases}$;
(f) $\theta(x) = \begin{cases} -1 & \text{if } x \leq -1 \\ 3x + 2 & \text{if } \mid x \mid < 1 \\ 7 - 2x & \text{if } x \geq 1 \end{cases}$; (g) $\phi(x) = \begin{cases} \sqrt{-x} & \text{if } x \leq 0 \\ x & \text{if } 0 \leq x \leq 2 \\ \sqrt{x - 2} & \text{if } x > 2 \end{cases}$;
(h) $f(x) = \frac{x}{|x|}$; $\qquad g(x) = \frac{x^2+5x+6}{x+2}$.

4. State whether the curve is the graph of a function of x. If it is, state domain and range of the function.

5. Find a function whose graph is the given curve :
(a) the line segment joining the points (-2,1) and (4,-6) ;
(b) the bottom half of the parabola $x + (y - 1)^2 = 0$;
(c) the top half of the circle $(x - 1)^2 + y^2 = 1$.

6. Find a formula for the described function and state its domain .
(a) A rectangle has perimeter $20\,m$. Express the area of the rectangle as a function of the length of one of its sides.
(b) A rectangle has area $16\,m^2$. Express the perimeter of the rectangle as a function of the length of one of its sides .
(c) Express the area of an equilateral triangle as a function of the length of a side .
(d) Express the surface area of a cube as a function of its volume .
(e) A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is $30\,ft$ express the area A of the window as a function of the width x of the window .
(f) 1.As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of $1\,km$ is 10°C, express the temperature T(in °C) as a function of the height h (in kilometers), assuming the function is linear.
2. Draw the graph of the function in part 1. What does the slope represent? What is the temperature at a height of $2,5\,km$?

7. Find the functions $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ and their domains :
(a) $f(x) = 2x^2 - x$, $g(x) = 3x + 2$; (b) $f(x) = \frac{1}{x}$, $g(x) = x^3 + 2x$;
(c) $f(x) = \frac{x+2}{2x+1}$, $g(x) = \frac{x}{x-2}$; $\qquad$ (d) $f(x) = \frac{1}{\sqrt{x}}$, $g(x) = x^2 - 4x$.

8. Find $f \circ g \circ h$ if :
(a) $f(x) = \frac{1}{x}$, $g(x) = x^3$, $h(x) = x^2 + 2$;
(b) $f(x) = x^4 + 1$, $g(x) = x - 5$, $h(x) = \sqrt{x}$.

9. (a)If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$;
(b) If $f(x) = x+4$ and $h(x) = 4x-1$, find a function g such that $g \circ f = h$.

10. Let $f(x) = \frac{1}{x}$ and $g(x) = x$. How does $f \circ f$ differ from g?