THE LIMIT OF A FUNCTION

Let us investigate the behavior of the function f defined by $f(x) = x^2 - x + 2$ for values near 2. The following table gives values of $f(x)$ for values of x close to 2 but not equal to 2.

x	$f(x)$	x	$f(x)$
$1,0$	$2,000000$	$3,0$	$8,000000$
$1,5$	$2,750000$	$2,5$	$5,750000$
$1,8$	$3,440000$	$2,2$	$4,640000$
$1,9$	$3,710000$	$2,1$	$4,310000$
$1,95$	$3,852500$	$2,05$	$4,152500$
$1,99$	$3,970100$	$2,01$	$4,030100$
$1,995$	$3,985025$	$2,005$	$4,015025$
$1,999$	$3,997001$	$2,001$	$4,003001$

From the table and the graph of f(a parabola) shown in Figure 1 we see that when x is close to 2 (on either side of 2) $f(x)$ is close to 4. In fact, it appears that we can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2. We express this by saying " the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2 is equal to 4." The notation for this is

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$

In general, we use the following notation.

DEFINITION 1 We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"
if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a but not equal to a.

Roughly speaking, this says that the values of $f(x)$ get closer and closer to the number L as x get closer and closer to the number a (from either side of a) but $x \neq a$. An alternative notation for

$$\lim_{x \to a} f(x) = L \quad \text{is} \qquad f(x) \to L \quad \text{as} \quad x \to a$$

which is usually read " $f(x)$ approaches L as x approaches a."
Notice the phrase "but $x \neq a$" in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined near a. Figure 2 shows the graphs of three functions. Note that

in part (c) $f(a)$ is not defined and in part (b), $f(a) \neq L$. But in each case, regardless of what happens at a, $\lim_{x \to a} f(x) = L$.

EXAMPLE Guess the value of $\lim_{x \to 1} \dfrac{x-1}{x^2-1}$.

SOLUTION Notice that the function $f(x) = \frac{x-1}{x^2-1}$ is not defined when $x = 1$, but that doesn't matter because the definition of $\lim_{x \to a} f(x)$ says that we consider values of x that are close to a but not equal to a. The following table gives values of $f(x)$ (correct to six decimal places) for values of x that approach 1 (but are not equal to 1).

x	$f(x)$
$0,5$	$0,666667$
$0,9$	$0,526316$
$0,99$	0.502513
$0,999$	$0,500250$
$0,9999$	$0,500025$

x	$f(x)$
$1,5$	$0,400000$
$1,1$	$0,476190$
$1,01$	0.497512
$1,001$	$0,499750$
$1,0001$	$0,499975$

On the basis of the values in the table, we make the guess that $\lim_{x \to 1} \dfrac{x-1}{x^2-1} = \dfrac{1}{2}$. This example is illustrated by the graph of f in Figure 3. Now let us change f slightly by giving it the value 2 when $x = 1$ and calling the resulting function g:

$$g(x) = \begin{cases} \dfrac{x-1}{x^2-1}, & \text{if} \quad x \neq 1 \\ 2, & \text{if} \quad x = 1 \end{cases}$$

This new function g still has the same limit as x approaches 1 (see Figure 4) .

EXAMPLE Find $\lim_{x \to 0} \dfrac{\sin x}{x}$.

SOLUTION Again the function $f(x) = \frac{\sin x}{x}$ is not defined when $x = 0$. Using a calculator (and remembering that, if $x \in \mathcal{R}$, $\sin x$ means the sine of the angle whose *radian* measure is x), we construct the accompanying table of values correct to eight decimal places. From the table and Figure 5 (drawn with the aid of the table) we guess that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

x	$\dfrac{\sin x}{x}$
± 1.0	0.84147098
± 0.5	0.95885108
± 0.4	0.97354586
± 0.3	0.98506736
± 0.2	0.99334665
± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	$0,99999983$

This guess is in fact correct, as will be proved in another Chapter using a geometric argument.

EXAMPLE Find $\lim_{x\to 0} \sin \frac{\pi}{x}$.
SOLUTION Once again the function $f(x) = \sin(\pi/x)$ is undefined at 0. Evaluating the function for some small values of x, we get

$f(1) = \sin \pi = 0$ $\quad f(\frac{1}{2}) = \sin 2\pi = 0$
$f(\frac{1}{3}) = \sin 3\pi = 0$ $\quad f(\frac{1}{4}) = \sin 4\pi = 0$
$f(0.1) = \sin 10\pi = 0$ $\quad f(0.01) = \sin 100\pi = 0$

Similarly, $f(0.001) = f(0.0001) = 0$. On the basis of this information we might be tempted to guess that

$$\lim_{x\to 0} \sin \frac{\pi}{x} = 0$$

but this time our guess is wrong. Note that although $f(\frac{1}{n}) = \sin n\pi = 0$ for any integer n, it is also true that $f(x) = 1$ for infinitely many values of x that approach 0. In fact , $\sin(\frac{\pi}{x}) = 1$ when $\frac{\pi}{x} = \frac{\pi}{2} + 2n\pi$ and solving for x, we get $x = \frac{2}{4n+1}$. The graph of f is given in Figure 6. The broken lines indicate that the values of $\sin(\frac{\pi}{x})$ oscillate between 1 and -1 infinitely often as x approaches 0. Since the values of $f(x)$ do not approach a fixed number as x approaches 0,

$$\lim_{x\to 0} \sin(\frac{\pi}{x}) \quad \text{does not exist.}$$

EXAMPLE The Heaviside function H is defined by $H(t) = \begin{cases} 0, & \text{if} \quad t < 0 \\ 1, & \text{if} \quad t \geq 0 \end{cases}$

This function is named after the electrical engineer Oliver Heaviside (1850-1925) and can be used to describe an electric current that is switched on at time $t = 0$. Its graph is shown in Figure 7.
As t approaches 0 from the left, $H(t)$ approaches 0. As t approaches 0 from the right, $H(t)$ approaches 1. There is no single number that $H(t)$ approaches as t approaches 0. Therefore $\lim_{t\to 0} H(t)$ does not exist.

ONE - SIDED LIMITS

In the last example we noticed that $H(t)$ approaches 0 as t approaches 0 from the left and $H(t)$ approaches 1 as t approaches 0 from the right. We indicate this situation symbolically by writing

$$\lim_{t\to 0^-} H(t) = 0 \qquad \text{and} \qquad \lim_{t\to 0^+} H(t) = 1$$

The symbol "$t \to 0^-$" indicates that we consider only values of t that are less than 0. Likewise "$t \to 0^+$" indicates that we consider only values of t that are greater than 0.

DEFINITION 2 We write $\lim_{x\to a^-} f(x) = L$ and say the left-hand limit of $f(x)$ as x approaches a a [or the limit of $f(x)$ as x approaches a from the left] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

Notice that Definition 2 differs from Definition 1 only in that we require x to be less than a. Similarly, if we require that x be greater than a, we get "the right-hand limit of $f(x)$ as x approaches is equal to L and we write

$$\lim_{x\to a^+} f(x) = L$$

Thus the symbol "$x \to a^+$" means that we consider only $x > a$.
By comparing Definition 1 with the definition of one-sided limits, we see that the following is true.

DEF. 3 $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$

EXAMPLE The graph of a function g is shown in Figure 8. Use it to state the values (if they exsist) of the following:
(a) $\lim_{x\to 2^-} g(x)$ (b) $\lim_{x\to 2^+} g(x)$ (c) $\lim_{x\to 2} g(x)$ (d) $g(2)$
(e) $\lim_{x\to 5^-} g(x)$ (f) $\lim_{x\to 5^+} g(x)$ (g) $\lim_{x\to 5} g(x)$ (h) $g(5)$

SOLUTION
From the graph we see that (a) $\lim_{x\to 2^-} g(x) = 3$ and (b) $\lim_{x\to 2^+} g(x) = 1$
(c) Since the left and right limits are different, we conclude that $\lim_{x\to 2} g(x)$ does not exist. (d) $g(2)$ does not exist.
The graph also shows that (e) $\lim_{x\to 5^-} g(x) = 2$ and (f) $\lim_{x\to 5^+} g(x) = 2$.
(g) $\lim_{x\to 5} g(x) = 2$. (h) Despite this fact, notice $g(5) \neq 2$ and $g(5) = 1$.

INFINITE LIMITS

EXAMPLE Find $\lim_{x\to 0} \frac{1}{x^2}$ if it is exists.

SOLUTION As x becomes close to 0, x^2 also becames close to 0, and $\frac{1}{x^2}$ becomes very large.(See the table.)

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10000
± 0.001	1000000

In fact, it appears from the graph of the function $f(x) = \frac{1}{x^2}$ shown in F.9 that the values of $f(x)$ can be made arbitrarily large by taking x close enough to 0. Thus the values of $f(x)$ do not approach a number , so $\lim_{x\to 0}(\frac{1}{x^2})$ does not exist. To indicate the kind of behavior exhibited in EXAMPLE we use the notation

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

This does not mean that we are regarding ∞ as a number. Nor does it mean that the limit exists. It simply expresses the particular way in which the limit does not exist: $\frac{1}{x^2}$ can be made as large as we like by taking x close enough to 0. In general, we write symbolically

$$\lim_{x\to a} f(x) = \infty$$

to indicate that the values of $f(x)$ become larger and larger (or "increase without bound") as x becomes closer and closer to a.

DEFINITION 4 Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a (but not equal to a).

Another notation for $\lim_{x\to a} f(x) = \infty$ is $f(x) \to \infty$ as $x \to a$
Again, the symbol ∞ is not number , but the expression $\lim_{x\to a} f(x) = \infty$ is often read as " the limit of $f(x)$, as x approaches a, is infinity"
or " $f(x)$ becomes infinite as x approaches a"
or " $f(x)$ increases without bound as x approaches a".(See Figure 10.)
A similar sort of of limit, for function that become large negative as x gets close to a, is defined in Definition 5.(See Figure 11.)

DEFINITION 5 Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative(as large as we please) by taking x sufficiently close to a (but not equal to a).

The symbol $\lim_{x\to a} f(x) = -\infty$ can be read as " the limit of $f(x)$, as x approaches a, is negative infinity" or " $f(x)$ decreases without bound as x approaches a."
As an example we have $\lim_{x\to 0}(-\frac{1}{x^2}) = -\infty$.
Similar definitions can be given for the one-sided infinite limits
$\lim_{x\to a^-} f(x) = \infty$ $\lim_{x\to a^+} f(x) = \infty$
$\lim_{x\to a^-} f(x) = -\infty$ $\lim_{x\to a^+} f(x) = -\infty$
remembering that "$x \to a^-$" means that we consider only values of x that are less then a, and similarly "$x \to a^+$" means that we consider only values of x that are larger then a.
Illustrations of these four cases are given in Figure 12.

EXERCISES

1. For the function f whose graph is given, state the value of the given quantity, if it exists.(Graph is given in Figure 13.)

$(a)\ \lim_{x\to 1} f(x)$	$(b)\ \lim_{x\to 3^-} f(x)$	$(c)\ \lim_{x\to 3^+} f(x)$
$(d)\ \lim_{x\to 3} f(x)$	$(e)\ f(3)$	$(f)\ \lim_{x\to -2^-} f(x)$
$(g)\ \lim_{x\to -2^+} f(x)$	$(h)\ \lim_{x\to -2} f(x)$	$(i)\ f(-2)$

2. For the function g whose graph is given, state the value of the given quantity, if it exists.(Graph is given in Figure 14.)

$(a)\ \lim_{x\to -2^-} g(x)$	$(b)\ \lim_{x\to -2^+} g(x)$	$(c)\ \lim_{x\to -2} g(x)$	$(d)\ g(-2)$
$(e)\ \lim_{x\to 2^-} g(x)$	$(f)\ \lim_{x\to 2^+} g(x)$	$(g)\ \lim_{x\to 2} g(x)$	$(h)\ g(2)$
$(i)\ \lim_{x\to 4^+} g(x)$	$(j)\ \lim_{x\to 4^-} g(x)$	$(k)\ g(0)$	$(l)\ \lim_{x\to 0} g(x)$

3. State the value of the limit, if it is exists, from the given graph.(Graph is given in Figure 15.)

$$(a) \lim_{x\to 3} f(x) \quad (b) \lim_{x\to 1} f(x) \quad (c) \lim_{x\to -3} f(x)$$
$$(d) \lim_{x\to 2^-} f(x) \quad (e) \lim_{x\to 2^+} f(x) \quad (f) \lim_{x\to 2} f(x)$$

4. A patient receives a $150-mg$ injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours.(Later we will be able to compute dosages and time intervals to ensure that the concentration of the drug does not reach harmful levels.)
Find $\lim_{t\to 12^-} f(t)$ and $\lim_{t\to 12^+} f(t)$ and explain the significance of these one-sided limits.(The graph is given in Figure16.)

5. (a) Sketch the graph of the function

$$f(x) = \begin{cases} 1-x & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(b) Use the graph from part (a) to state the value of each of the following limits, if it exists.
(i) $\lim_{x\to 0^-} f(x)$ (ii) $\lim_{x\to 0^+} f(x)$ (iii) $\lim_{x\to 0} f(x)$

6. (a) Sketch the graph of the function

$$f(x) = \begin{cases} 2-x & \text{if } x < -1 \\ x & \text{if } -1 \le x < 1 \\ 4 & \text{if } x = 1 \\ 4-x & \text{if } x > 1 \end{cases}$$

(b) Use the graph from part (a) to state the value of each of the following limits, if it exists.

$$(i) \lim_{x\to -1^-} g(x) \quad (ii) \lim_{x\to -1^+} g(x) \quad (iii) \lim_{x\to -1} g(x)$$
$$(iv) \lim_{x\to 1^-} g(x) \quad (v) \lim_{x\to 1^+} g(x) \quad (vi) \lim_{x\to 1} g(x)$$

7. Determine the infinite limit:

$$(a) \lim_{x\to 5^+} \frac{6}{x-5} \quad (b) \lim_{x\to 5^-} \frac{6}{x-5} \quad (c) \lim_{x\to 3} \frac{1}{(x-3)^8}$$
$$(d) \lim_{x\to 0} \frac{x-1}{x^2(x+2)} \quad (e) \lim_{x\to -2^+} \frac{x-1}{x^2(x+2)} \quad (f) \lim_{x\to \pi^-} \frac{1}{\sin x}$$